Polynomial and Rational Functions Key Concepts

Polynomial Functions Lesson

Polynomial Function

A polynomial function is any function of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$ where n is a nonzero whole number, and the coefficients $a_0 \ldots a_n$ are real numbers.

The largest exponent, n, is the degree of the polynomial.

a_n is called the leading coefficient and is not equal to zero.

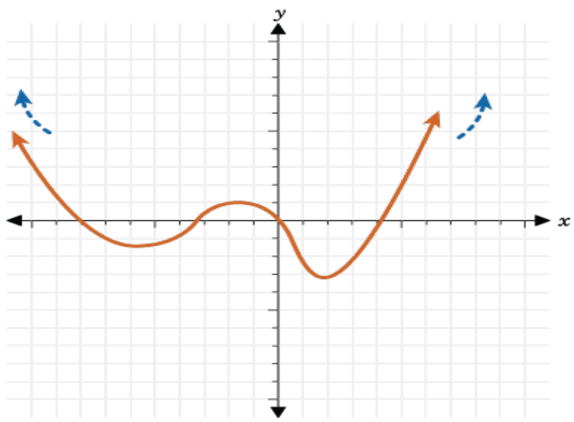
The domain for polynomial functions is the set of all real numbers, $(-\infty, \infty)$.

Leading Term Test

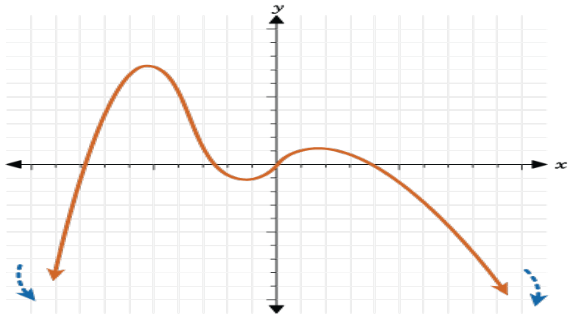
For a polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, the end behavior of the graph depends on the degree of the polynomial, n, and the sign of the leading coefficient, a_n.

Even-Degree Functions

- Even-degree functions behave in the same way at both ends of the graph.
- If the degree of a polynomial function is even and the leading coefficient, a_n, is positive, $f(x) \to \infty$ both as $x \to \infty$ and as $x \to -\infty$.
- The graph will rise on both the left and right ends.

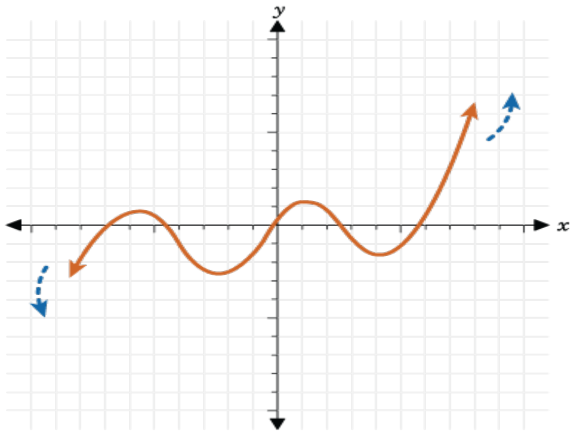


- If the degree of a polynomial function is even and the leading coefficient, a_n, is negative, $f(x) \to -\infty$ both as $x \to \infty$ and as $x \to -\infty$.
- The graph will fall on both the left and right ends.

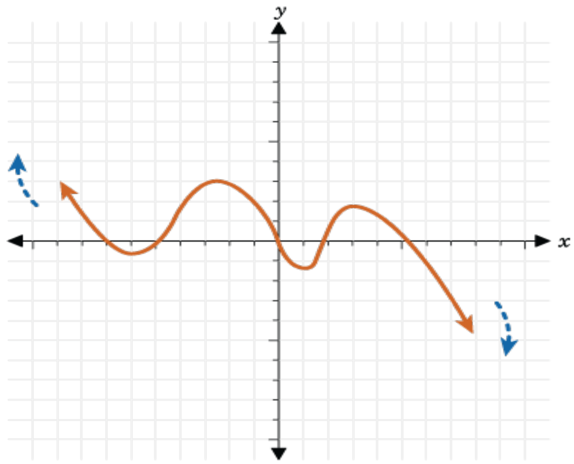


Odd-Degree Functions

- Odd-degree functions behave in opposite ways at each end of the graph.
- If the degree of a polynomial function is odd and the leading coefficient, a_n, is positive, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$.
- The graph will rise to the right and fall to the left.



- If the degree of a polynomial function is odd and the leading coefficient, a_n, is negative, $f(x) \to -\infty$ as $x \to \infty$ and $f(x) \to \infty$ as $x \to -\infty$.
- The graph will fall to the right and rise to the left.



Global and Local Extrema

A function f has a global maximum of $f(c)$ if $f(c) \geq f(x)$ for all x in the domain of the function.

A function f has a global minimum of $f(c)$ if $f(c) \leq f(x)$ for all x in the domain of the function.

A function f has a local maximum of $f(c)$ if $f(c) \geq f(x)$ for all x in the open interval (a,b) that contains c.

A function f has a local minimum of $f(c)$ if $f(c) \leq f(x)$ for all x in the open interval (a,b) that contains c.

Extreme Value Theorem

If a function f is continuous on a closed interval $[a,b]$, then f has a maximum and minimum on that interval.

Real Zeroes of a Polynomial Function Lesson

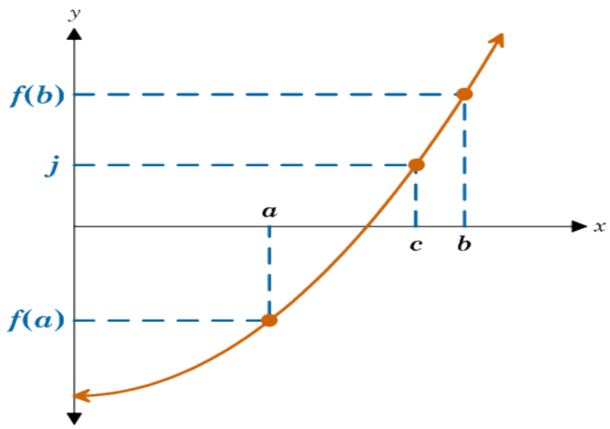
Identifying Multiplicity from the Graph of a Polynomial Function

If the multiplicity of a zero is odd, then the graph will cross the *x*-axis at that zero.

If the multiplicity of a zero is even, the graph will touch, but not cross, the *x*-axis at that zero.

Intermediate Value Theorem

Let $f(x)$ be a polynomial with real coefficients. For any value, j, between $f(a)$ and $f(b)$, there exists at least one value *c* in the interval (a,b) such that $f(c)=j$.



Descartes's Rule of Signs

Let *f* be a polynomial function with real coefficients.

- The number of positive real zeroes of $f(x)$ is either equal to the number of sign changes of $f(x)$ or is less than the number of sign changes by an even integer.
- The number of negative real zeroes of $f(x)$ is either equal to the number of sign changes of $f(-x)$ or is less than the number of sign changes by an even integer.
- If $f(x)$ has one sign change, there will be one positive real zero.
- If $f(-x)$ has one sign change, there will be one negative real zero.

Rational Zero Theorem

If the polynomial function $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x^1 + a_0$ has integer coefficients, then every rational zero has the form $\frac{p}{q}$, where *p* is a factor of the constant term a_0 and *q* is a factor of the leading coefficient a_n.

Dividing Polynomials Lesson

The Division Algorithm

If $f(x)$ and $D(x)$ are polynomial functions, where $D(x) \neq 0$ and $D(x)$ is of lesser or equal degree than $f(x)$, then there exist polynomials $Q(x)$ and $R(x)$ such that $f(x) = D(x) \cdot Q(x) + R(x)$, where the degree of $R(x)$ is 0 or of a lesser degree than $D(x)$.

- $f(x)$ is the dividend.
- $D(x)$ is the divisor.
- $Q(x)$ is the quotient.
- $R(x)$ is the remainder.

If $R(x) = 0$, then $D(x)$ and $Q(x)$ are factors of $f(x)$.

The Remainder Theorem

If a polynomial $f(x)$ is divided by $x - a$, then the remainder is $f(a)$.

Using the division algorithm, $f(x) = (x - a)Q(x) + f(a)$.

The Factor Theorem

If $f(x)$ is a polynomial function, then $x - a$ is a factor of $f(x)$ if and only if $f(a) = 0$.

Identifying Multiplicity from the Factors of a Polynomial

If $(x - a)^k$ is a factor of polynomial *f*, then *k* is the multiplicity of the zero *a*.

Complex Zeroes of a Polynomial Function Lesson

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial function with rational coefficients and degree *n*, where $n > 0$, then $f(x)$ has exactly *n* roots.

Conjugates Zero Theorem

Let $f(x)$ be a polynomial function with real coefficients. If $z = a + bi$ where $b \neq 0$ is a zero of $f(x)$, then the complex conjugate $\overline{z} = a - bi$ is also a zero of $f(x)$.

Let $f(x)$ be a polynomial function of odd degree with real coefficients; then $f(x)$ has at least one real root.

Graphs of Rational Functions Lesson

Rational Function

A rational function *f* is the ratio of two polynomial functions $p(x)$ to $q(x)$, where the function in the denominator, $q(x)$, is not the zero function.

The domain of a rational function is the set of all real numbers such that $q(x) \neq 0$.

Vertical Asymptotes

A vertical asymptote at $x = c$ for the graph of *f* is a vertical line whose distance from the graph of the function nears zero as the value of the independent variable approaches *c*, denoted $x \to c$.

Removable Discontinuity

If the graph of a function has a removable discontinuity at $x = a$, there will be a hole in the graph of the function at this point. There is no vertical asymptote at this location.

Horizontal Asymptotes

A horizontal asymptote for the graph of a rational function is a horizontal line for which the distance from the graph of the function to the line approaches zero as the value of the independent variable approaches ∞ or $-\infty$.

Finding Horizontal Asymptotes

The rational function $\boldsymbol{f(x) = \frac{p(x)}{q(x)}}$, where the degree of $p(x)$ is *n* and the degree of $q(x)$ is *m*, has the following characteristics:

- If $n < m$, then the function has a horizontal asymptote at *y* = 0.
- If *n* = *m*, then the function has a horizontal asymptote at $y = \frac{a}{b}$, where *a* is the leading coefficient of *p* and *b* is the leading coefficient of *q*.
- If $n > m$, then there is no horizontal asymptote.

Finding Slant Asymptotes

Given the rational function $f(x) = \frac{p(x)}{q(x)}$, where the degree of $p(x)$ is *n* and the degree of $q(x)$ is *m*, if *n* = *m* + 1 then the function has a slant asymptote. The equation for the slant asymptote is equal to the quotient of $p(x)$ and $q(x)$, without the remainder.

Operations with Rational Expressions Lesson

Adding and Subtracting Rational Expressions

To add or subtract rational expressions, rewrite the expressions with a common denominator if necessary. Then add the expressions in the numerator, and keep the common denominator.

Multiplying Rational Expressions

To multiply rational expressions, multiply the numerators and multiply the denominators. Then simplify the resulting rational expression.

Dividing Rational Expressions

To divide rational expressions, multiply the first rational expression by the reciprocal of the second rational expression. Then, factor the numerator and denominator to simplify.