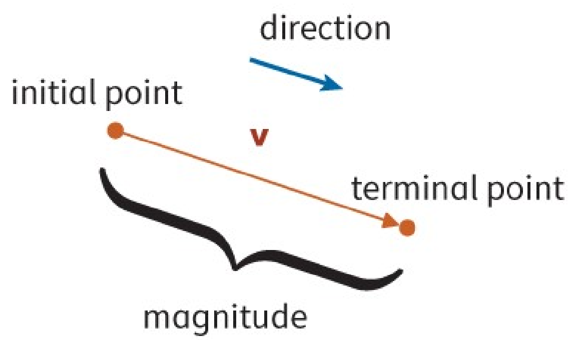
UNIT 5

PRECALCULUS B

LESSONS:

- Representing Vectors
- Operations with Vectors
- Unit Vectors
- Direction Angle
- Dot Product
- Angle Between Two Vectors

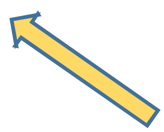
What is a vector?



What is a vector?

VECTOR: I'm applying for a villain loan. I go by Vector. It's a mathematical term, represented by an arrow with both direction and magnitude. Vector! That's me, because I commit crimes with both direction and magnitude. Oh yeah!

From "Despicable Me"



What is the notation for writing a vector?

3 Notation Types:

- **Component form**
- **Linear Combination form**
- **Magnitude & Angle of Direction form**

COMPONENT FORM: **v** = < v_1 , v_2 >

v = <v_1, v_2> with v_1 as the horizontal component and v_2 as the vertical component

v = <horizontal distance from initial to terminal, vertical distance from initial to terminal>

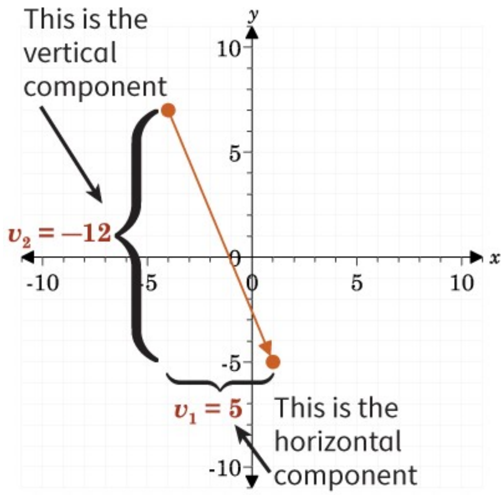
v = <x-coordinate of terminal – x-coordinate of initial, y-coordinate of terminal – y-coordinate of initial>

If the initial point is P with coordinates (p_1, p_2)
and the terminal point is Q with coordinates (q_1, q_2)

Then, the vector **v** = <$q_1 - p_1$, $q_2 - p_2$>

which is then **v** = <v_1, v_2>

COMPONENT FORM: $\mathbf{v} = \langle v_1 , v_2 \rangle$



$\mathbf{v} = \langle v_1, v_2 \rangle$

Initial point P (-4, 7)
& Terminal point Q (1, -5)

Then, the vector $\mathbf{v} = \langle q_1 - p_1, q_2 - p_2 \rangle$

Is $\langle 1 - -4, -5 - 7 \rangle$

Which is $\mathbf{v} = \langle 5, -12 \rangle$

For <right 5, down 12>

COMPONENT FORM: $\mathbf{v} = \langle v_1 , v_2 \rangle$

TRY IT:

Write the vector for an initial point of (-5, 4) and a terminal point of (4, 8)

Write the vector for an initial point of (2, -3) and a terminal point of (-4, -5)

COMPONENT FORM: $\mathbf{v} = \langle v_1 , v_2 \rangle$

TRY IT:

Write the vector for an initial point of (-5, 4) and a terminal point of (4, 8)

v = <9, 4>

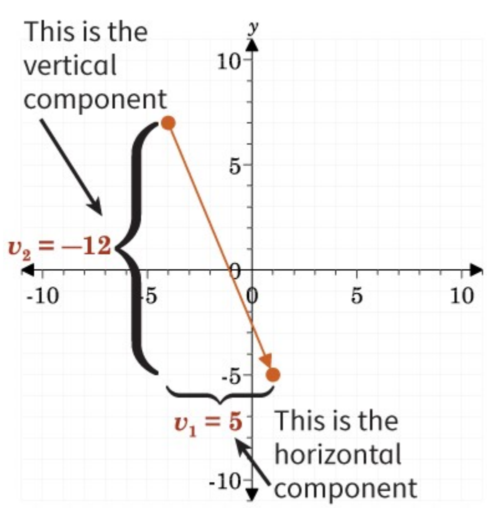
… for 9 right & 4 up

Write the vector for an initial point of (2, -3) and a terminal point of (-4, -5)

v = <-6, -2>

… for 6 left & 2 down

MAGNITUDE (Length) of a Vector: | | v | |



$$\|\mathbf{v}\| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2} = \sqrt{(v_1)^2 + (v_2)^2}$$

v = <5, -12>

$$\|\mathbf{v}\| = \sqrt{(5)^2 + (-12)^2} = \sqrt{169} = 13$$

***Yes, this is using the Pythagorean Theorem with the horizontal & vertical components as the sides of right triangle, and the hypotenuse is the length of the vector!**

MAGNITUDE (Length) of a Vector: $||v||$

TRY IT:

Calculate the magnitude of
v = <9, 4>

Calculate the magnitude of
v = <-6, -2>

MAGNITUDE (Length) of a Vector: $||v||$

TRY IT:

Calculate the magnitude of
v = <9, 4>

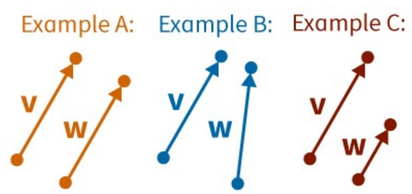
$||v|| = \sqrt{9^2 + 4^2} = \sqrt{81+16} = \sqrt{97}$

Calculate the magnitude of
v = <-6, -2>

$||v|| = \sqrt{-6^2 + -2^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

MAGNITUDE (Length) of a Vector: | | v | |

NOTE:
Two vectors are equal if they have the same magnitude and direction.
They don't have to be at the same point coordinates!

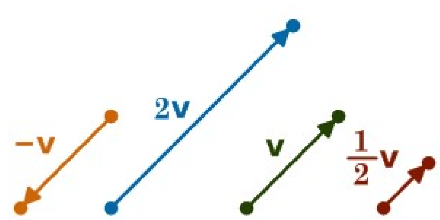


A: same magnitude, same direction
B: same magnitude, different direction
C: different magnitude, same direction

MAGNITUDE (Length) of a Vector: | | v | |

SCALAR MULTIPLICATION

A vector can have its magnitude (length) changed by multiplying by a single number called a SCALAR.



So, for v = <2, -4>

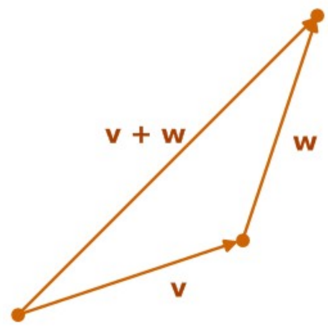
3v = <3 · 2, 3 · -4>

3v = <6, -12>

ADDITION of Vectors:

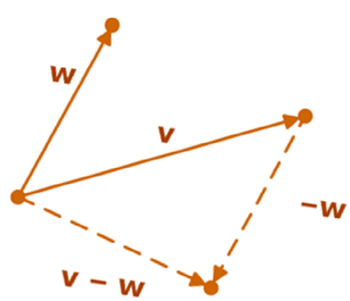
To add two vectors, start the second vector at the end of the first vector, then the sum vector goes from the initial point of the first vector to the terminal point of the second vector.

The final vector is called the "resultant vector".



SUBTRACTION of Vectors:

To subtract two vectors, reverse the direction of the second vector, then put its new initial point at the end of the first vector, and the final vector will go from the new terminal point of the second vector back to the initial point of the first vector.



ADDITION and SUBTRACTION of Vectors:

Algebraically, just add or subtract the components!

TRY IT:

If v = <2, 5> and w = <4, -3>

Then v + w = ?

And v – w = ?

ADDITION and SUBTRACTION of Vectors:

Algebraically, just add or subtract the components!

TRY IT:

If v = <2, 5> and w = <4, -3>

Then v + w = < 2 + 4, 5 + -3 > = < 6, 2 >

And v – w = < 2 – 4, 5 - -3 > = < -2, 8 >

NOW TRY:

3v + 2w

ADDITION and SUBTRACTION of Vectors:

Algebraically, just add or subtract the components!

TRY IT:

If v = <2, 5> and w = <4, -3>

Then v + w = < 2 + 4, 5 + -3 > = < 6, 2 >

And v – w = < 2 – 4, 5 - -3 > = < -2, 8 >

NOW TRY:

3v + 2w

3<2,5> + 2<4,-3>

<6,15> + <8,-6>

<6+8, 15+-6>

<14, 9>

LINEAR COMBINATION FORM: $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}$

i is the horizontal standard unit vector (a horizontal vector of length 1 unit)

j is the vertical standard unit vector (a vertical vector of length 1 unit)

Because, each component can be thought of as being made up of multiple single unit vectors!

And then added because the final vector is the sum of the horizontal and vertical vectors!

So, $\mathbf{v} = \langle v_1, v_2 \rangle$

can also be written as $\mathbf{v} = v_1\,\mathbf{i} + v_2\,\mathbf{j}$

And this form works the same way as the component form with addition, subtraction, and scalar multiplication

UNIT VECTORS: $\mathbf{u} = \dfrac{\mathbf{v}}{||\mathbf{v}||}$

But you can have a 1 unit vector in <u>any</u> direction, not just horizontal or vertical!

For example, a vector with a magnitude of 5 could be cut into 5 one unit vectors.

v = <3, 4> has a magnitude of 5

So, divide each component by 5 to get the components of the 1unit long vector in the same direction!

u = <3/5, 4/5> and has a magnitude of 1

MAGNITUDE & ANGLE OF DIRECTION FORM:

$$\mathbf{v} = \langle\, ||\mathbf{v}|| \cos\theta,\ ||\mathbf{v}|| \sin\theta \,\rangle$$

or

$$\mathbf{v} = ||\mathbf{v}|| \cos\theta\, \mathbf{i} + ||\mathbf{v}|| \sin\theta\, \mathbf{j}$$

This is like in polar coordinates where you use the length and the angle.

The angle θ is the direction angle, that is, the counter-clockwise angle from the horizontal base.

We get this from setting the initial point of the vector at the origin and calculating the reference angle α (alpha).

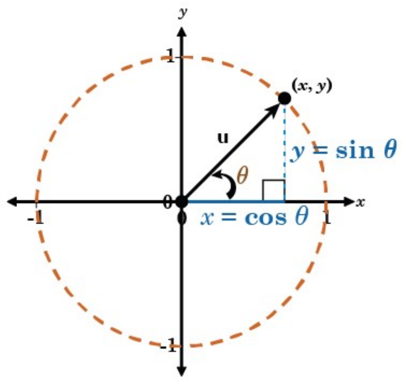
Then, check which quadrant the terminal point is in for determining the actual direction angle θ (theta).

MAGNITUDE & ANGLE OF DIRECTION FORM:

$$\mathbf{v} = \langle \|\mathbf{v}\| \cos\theta, \|\mathbf{v}\| \sin\theta \rangle$$

or

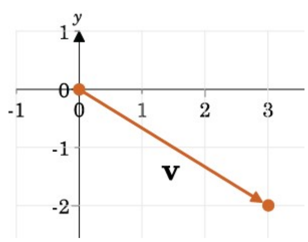
$$\mathbf{v} = \|\mathbf{v}\| \cos\theta \, \mathbf{i} + \|\mathbf{v}\| \sin\theta \, \mathbf{j}$$



Since tangent = opposite/adjacent, and the vertical component is opposite and the horizontal component is adjacent, we can find the reference angle with:

$$\tan\alpha = \left|\frac{v_2}{v_1}\right|$$

ANGLE OF DIRECTION θ:



$$\tan\alpha = \left|\frac{v_2}{v_1}\right|$$

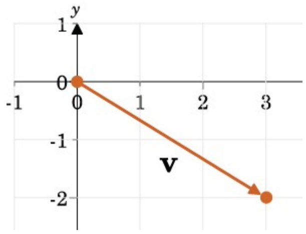
TRY IT: **$\langle 3, -2 \rangle$**

What is the reference angle?

What is the direction angle?

How do you write the vector in magnitude & angle form?

ANGLE OF DIRECTION θ:



$$\tan \alpha = \left|\frac{v_2}{v_1}\right|$$

TRY IT: <3, -2>

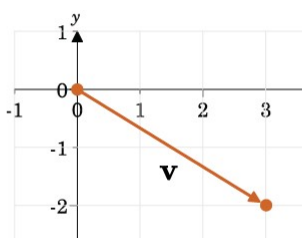
What is the reference angle?

tan a = |-2/3|
a = arctan |-2/3|
a ≈ 33.7°

What is the direction angle?

How do you write the vector in magnitude & angle form?

ANGLE OF DIRECTION θ:



$$\tan \alpha = \left|\frac{v_2}{v_1}\right|$$

TRY IT: <3, -2>

What is the reference angle?

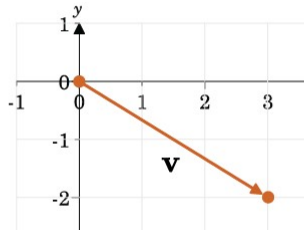
a ≈ 33.7°

What is the direction angle?

The vector is in Quadrant IV.
So, the counter-clockwise direction angle θ is
360 – 33.7 ≈ 326.3°

How do you write the vector in magnitude & angle form?

ANGLE OF DIRECTION θ:



$\tan \alpha = \left|\frac{v_2}{v_1}\right|$

TRY IT: <3, -2>

What is the reference angle?

$\alpha \approx 33.7°$

What is the direction angle?

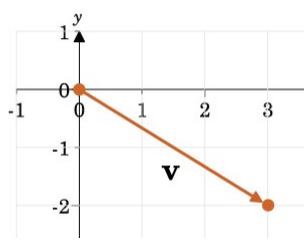
$\theta \approx 326.3°$

How do you write the vector in magnitude & angle form?

First, calculate the magnitude:

$$\|\mathbf{v}\| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$$

ANGLE OF DIRECTION θ:



$\tan \alpha = \left|\frac{v_2}{v_1}\right|$

TRY IT: <3, -2>

What is the reference angle?

$\alpha \approx 33.7°$

What is the direction angle?

$\theta \approx 326.3°$

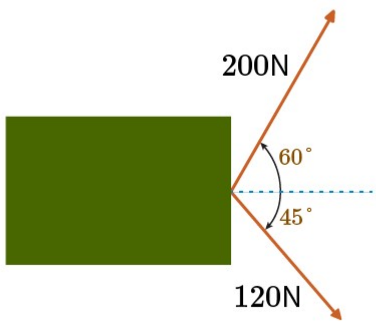
How do you write the vector in magnitude & angle form?

First, calculate the magnitude:

$\mathbf{v} = \langle \sqrt{13} \cos 326.3°, \sqrt{13} \sin 326.3° \rangle$

$\mathbf{v} \approx \langle 3, -2 \rangle$ OR $\mathbf{v} \approx 3\mathbf{i} - 2\mathbf{j}$

ADDING VECTORS using MAGNITUDE & ANGLE:



Hmm … the top force is stronger. I bet the box will get pulled more toward the top as it gets pulled forward.

TRY IT:

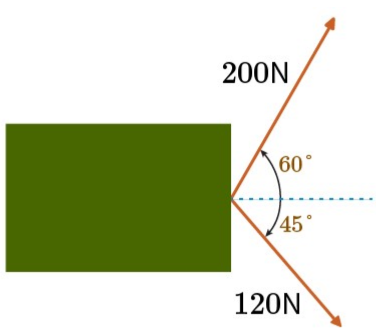
Calculate the components of each vector.

Add them to get the resultant vector.

Calculate the magnitude for the resultant force.

Calculate the resultant's direction angle.

ADDING VECTORS using MAGNITUDE & ANGLE:



TRY IT:

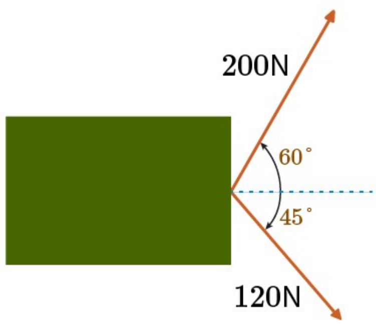
Calculate the components of each vector.

$\mathbf{v} = < ||\mathbf{v}|| \cos\theta,\ ||\mathbf{v}|| \sin\theta >$

$\mathbf{a} = < 200(\cos 60),\ 200(\sin 60) >$
$\mathbf{a} = <100,\ 173.2>$

$\mathbf{b} = < 120(\cos(-45)),\ 120(\sin(-45)) >$
$\mathbf{b} = <84.9,\ -84.9>$

ADDING VECTORS using MAGNITUDE & ANGLE:



TRY IT:

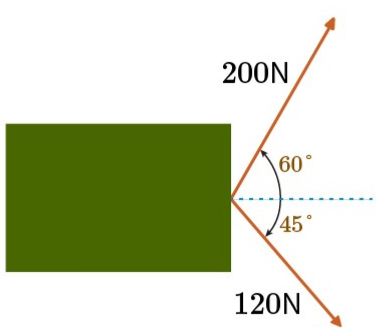
Add them to get the resultant vector.

a = <100, 173.2>
b = <84.9, -84.9>

a + **b** = <100 + 84.9, 173.2 - 84.9>

a + **b** = < 184.9, 88.3 >

ADDING VECTORS using MAGNITUDE & ANGLE:



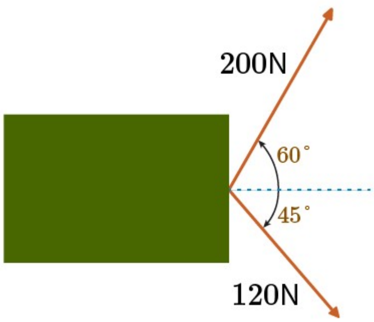
TRY IT:

Calculate the magnitude for the resultant force.

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{184.9^2 + 88.3^2} = 204.9 \text{ N.}$$

The resultant force is a little bit more than the stronger of the two original forces.

ADDING VECTORS using MAGNITUDE & ANGLE:



TRY IT:

Calculate the resultant's direction angle.

$$\theta = \arctan\left|\frac{88.3}{184.9}\right| = 25.5°$$

Yep! ... the box did get pulled more toward the top as it gets pulled forward.

What about the angle between ANY two vectors??

We can do that, but we need another tool . . .

DOT PRODUCT

$$\mathbf{u} \cdot \mathbf{v} = a_1a_2 + b_1b_2$$

$$\mathbf{u} = \langle a_1, b_1 \rangle \text{ and } \mathbf{v} = \langle a_2, b_2 \rangle$$

NOTE: The result is just a number, not a vector!

"The **dot product** is a vector operation that represents the sum of the products of the horizontal and vertical components of two vectors and results in a scalar."

DOT PRODUCT

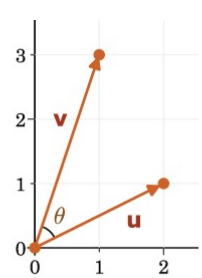
$$\mathbf{u} \cdot \mathbf{v} = a_1 a_2 + b_1 b_2$$

$$\mathbf{u} = \langle a_1, b_1 \rangle \text{ and } \mathbf{v} = \langle a_2, b_2 \rangle$$

This sentence from the lesson should make more sense now.

THE ANGLE BETWEEN ANY TWO VECTORS:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



TRY IT:

What are the components of the vectors?

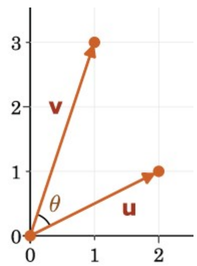
Find the Dot Product for the numerator.

Find the magnitude of each vector, then multiply for the denominator.

Divide, then calculate the arccosine.

THE ANGLE BETWEEN ANY TWO VECTORS:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



TRY IT:

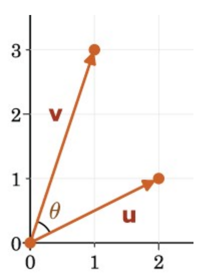
What are the components of the vectors?

u = <2, 1>   v = <1, 3>

Find the Dot Product for the numerator.
Find the magnitude of each vector, then multiply for the denominator.
Divide, then calculate the arccosine.

THE ANGLE BETWEEN ANY TWO VECTORS:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



TRY IT:

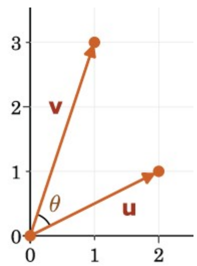
What are the components of the vectors?
u = <2, 1>   v = <1, 3>

Find the Dot Product for the numerator.
u · v = (2 · 1) + (1 · 3) = 5

Find the magnitude of each vector, then multiply for the denominator.
Divide, then calculate the arccosine.

THE ANGLE BETWEEN ANY TWO VECTORS:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



TRY IT:

What are the components of the vectors?
u = <2, 1>   v = <1, 3>

Find the Dot Product for the numerator.
u · v = (2 · 1) + (1 · 3) = 5

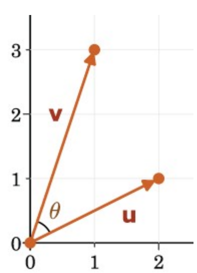
Find the magnitude of each vector, then multiply for the denominator.

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

Divide, then calculate the arccosine.

THE ANGLE BETWEEN ANY TWO VECTORS:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



TRY IT:

What are the components of the vectors?
u = <2, 1>   v = <1, 3>

Find the Dot Product for the numerator.
u · v = (2 · 1) + (1 · 3) = 5

Find the magnitude of each vector, then multiply for the denominator.

$$\|\mathbf{u}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\|\mathbf{v}\| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

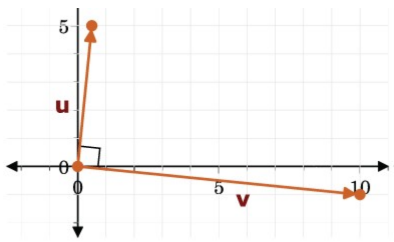
Divide, then calculate the arccosine.

$$\cos\theta = \frac{5}{\sqrt{5}\left(\sqrt{10}\right)}$$

$$\theta = 45°$$

THE ANGLE BETWEEN ANY TWO VECTORS – SPECIAL CASES:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$



What about when the vectors are perpendicular?

Actually, lines are perpendicular.
When the angle between vectors is 90°, they are called ORTHOGONAL vectors.

So, if θ is 90° . . . Then cos θ = 0

Which means the Dot Product = 0

Test for orthogonal vectors:
Does the Dot Product = 0?

THE ANGLE BETWEEN ANY TWO VECTORS – SPECIAL CASES:

$$\cos\theta = \frac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$$

What about when the vectors are parallel?

Parallel means there is no angle between the vectors.
And no angle means an angle of 0!

So, if θ is 0° . . . Then cos θ = 1 or -1

Which means the numerator = the denominator.

Test for parallel vectors:
Does the angle θ = 0?

One last application of vectors – WORK:

These are concepts from Physics.

Vectors can measure things like distance, force, and velocity.

"Work" is the result of force over a distance.

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos\theta$$

There are two approaches to calculating Work.
Use whichever matches the information given you.

If you know the components, do the dot product.
If you know the magnitudes and the angle, use the second option.

One last application of vectors – WORK:

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos\theta$$

This graphic gives the amount of the force in pounds.
This is the magnitude of the force.

This graphic also gives the angle.

TRY IT: Calculate the Work if the distance moved is 175 feet.

One last application of vectors – WORK:

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos\theta$$

$$W = (38\text{ lbs.})(175\text{ ft.})\cos 28^\circ$$
$$\approx 5{,}871.6\text{ ft}\cdot\text{lbs}$$

This graphic gives the amount of the force in pounds.
This is the magnitude of the force.

This graphic also gives the angle.

TRY IT: Calculate the Work if the distance moved is 175 feet.

One last application of vectors – WORK:

Find the work done by a 10-lb. force acting in the direction $\langle 1, 3\rangle$ in moving an object 5 ft. from $(0, 0)$ to $(5, 0)$. Round your answer to the nearest tenth of a foot-pound.

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\| \|\overrightarrow{PQ}\| \cos\theta$$

This gives the end points to calculate the components of the movement from P to Q.

BUT, the <1, 3> are NOT the components for the force vector!!

We can use them to calculate what we need . . .

One last application of vectors – WORK:

Find the work done by a 10-lb. force acting in the direction $\langle 1, 3\rangle$ in moving an object 5 ft. from $(0, 0)$ to $(5, 0)$. Round your answer to the nearest tenth of a foot-pound.

$$W = \mathbf{F} \cdot \overrightarrow{PQ} = \|\mathbf{F}\|\|\overrightarrow{PQ}\|\cos\theta$$

The force is "in the direction" of <1, 3> with a magnitude of 10 pounds.

We will calculate the magnitude of <1, 3>
Then divide each component by that magnitude to get the components of a vector in that direction that will now be only 1 unit long.

Multiply that unit vector for that direction by the force of 10, and we will have the components for this force vector.

One last application of vectors – WORK:

The force is "in the direction" of <1, 3> with a magnitude of 10 pounds.

We will calculate the magnitude of <1, 3>
Then divide each component by that magnitude to get the components of a vector in that direction that will now be only 1 unit long.

Multiply that unit vector for that direction by the force of 10, and we will have the components for this force vector.

$$\mathbf{F} = 10\left(\frac{\langle 1, 3\rangle}{\|\langle 1, 3\rangle\|}\right)$$

$$\mathbf{F} = 10\left(\frac{\langle 1, 3\rangle}{\sqrt{10}}\right)$$

$$\mathbf{F} = \left\langle \frac{10}{\sqrt{10}}, \frac{30}{\sqrt{10}} \right\rangle$$

One last application of vectors – WORK:

Find the work done by a 10-lb. force acting in the direction $\langle 1, 3\rangle$ in moving an object 5 ft. from $(0, 0)$ to $(5, 0)$. Round your answer to the nearest tenth of a foot-pound.

$$W = \mathbf{F} \bullet \overrightarrow{PQ} = \|\mathbf{F}\|\left\|\overrightarrow{PQ}\right\|\cos\theta$$

Now we have the components for the force and the movement to do the dot product!

$$W = F \bullet \overrightarrow{PQ}$$

$$W = \left\langle \frac{10}{\sqrt{10}}, \frac{30}{\sqrt{10}} \right\rangle \bullet \langle 5, 0\rangle$$

$$W = \left(\frac{10}{\sqrt{10}} \cdot 5\right) + \left(\frac{30}{\sqrt{10}} \cdot 0\right)$$

$$W = \frac{50}{\sqrt{10}}$$

$$W \approx 15.8 \text{ft} \cdot \text{lbs}$$

We have now added to your "Detective Manual":

- Vectors
- Magnitude
- Scalars
- Resultant vectors
- Vector components
- Linear combinations
- Unit vectors
- Direction Angle
- Dot product
- Orthogonal test
- Parallel test
- Work formulas